Notes on What Priors to Use, When, and Why, to Compare Asset Pricing Models in General

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This article provides an extensive discussion on what priors to use, when, and why, to compare asset pricing models in general. To be precise, I layout rules to derive priors under three different cases. The first case relates to Barillas and Shanken (2020) (BS henceforth), and Chib, Zeng, and Zhao (2020) (CZZ henceforth), which involves comparing multiple models that exclusively comprise traded factors. In the second, I discuss rules to test an individual model containing non-traded factors. The third considers comparing multiple models involving non-traded factors. I borrow the standard notations from the main section of the paper to represent variables and parameters such as the included and excluded factors, and regression coefficients in the excluded and included factor regressions, etc.,.

I. Priors for Comparing Models with Traded Factors

Notations: Let R denote the set of N test-assets. Let F_j, F_j^* denote the set of factors included and excluded in the model M_j, respectively. Let K_j and K_j^* denote the number of M_j's included and excluded factors, respectively.

To compare models that exclusively comprise traded factors, BS derive an important one-one mapping between the nuisance parameters of one model to the nuisance parameters of any other. Chib et al. (2020) (CZZ, henceforth) use this identity and derive modified Jeffreys (1998) priors that permit (marginal) likelihood-based model comparisons. In the BS-CZZ framework, the parameters of any model are induced using the product of two independent Jeffreys (1998) that correspond to

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the following regressions

$$R = FB_r + \epsilon_r,\ \epsilon_r \sim MVN(0, \Sigma_r \otimes I), \tag{1}$$
$$F = \alpha + \epsilon,\ \epsilon \sim MVN(0, \Sigma \otimes I). \tag{2}$$

In particular, M_j holds whenever

$$R = FB_r + \epsilon_r,\ \epsilon_r \sim MVN(0, \Sigma_r \otimes I), \tag{3}$$
$$F_j^* = F_j B_j^* + \epsilon_j^*,\ \epsilon_j^* \sim MVN(0, \Sigma_j^* \otimes I), \tag{4}$$
$$F_j = \alpha_j + \epsilon_j,\ \epsilon \sim MVN(0, \Sigma_j \otimes I). \tag{5}$$

Then, the BS-CZZ priors for the parameters under model M_j are given by

$$P(\Sigma_r, \Sigma_j, \Sigma_j^*) = \left(|\Sigma_r|^{-\frac{N+1}{2}}\right) \times \left(|\Sigma_j^*|^{\frac{K^T+1}{2}} |\Sigma_j|^{-\frac{2K_j - K^T+1}{2}}\right), \tag{6}$$

where Σ_r, Σ_j^*, Σ_j represent the residual covariance parameters in the test-assets, excluded and included factors regressions, respectively. As BS note, the priors in (6) yield *consistent* and *invariant* model comparisons.

Note from (6) that the BS-CZZ priors could be expressed as the product of two independent Jeffreys (1998), where the first involves test assets' regression parameters ((3)), and the second is induced from the all-factors regression ((4), (5)). Because of this independence specification, test assets dropout while comparing asset pricing models with traded factors.

However, BS and CZZ do not explicitly discuss why such an independence restriction must be imposed apriori between the parameters of the test asset and all factor regressions. Alternatively, one could induce the priors for the parameters under each model using a single Jeffreys (1998) that corresponds to the joint regression of the test assets and traded factors given by

$$[R, F] = \alpha_{all} + \epsilon_{all},\ \epsilon_{all} = MVN(0, \Sigma_{all} \otimes I), \tag{7}$$

where the priors are given by

$$P(\Sigma_{all}) = |\Sigma_{all}|^{-\frac{N+K^T+1}{2}}. \tag{8}$$

In this framework, M_j holds whenever

$$[R, F_j^*] = F_j B_j^* + \epsilon_j^*,\ \epsilon_j^* \sim MVN(0, \Sigma_j^* \otimes I), \tag{9}$$
$$F_j = \alpha_j + \epsilon_j,\ \epsilon \sim MVN(0, \Sigma_j \otimes I). \tag{10}$$

Surprisingly, it turns out that the priors for the parameters (in (9), (10)) induced from (8) also deliver marginal likelihoods that share the same marginalization constant across the models. Thus, in the spirit of CZZ, these MLs should deliver valid model comparisons. Similarly, these MLs satisfy the invariant and consistency properties of BS. Thus, a natural question that arises is, which prior specification between the original BS-CZZ ((6)) and the alternative specification ((8)) should one use?

I argue that one should always impose the *independence* restriction apriori and use the original BS-CZZ prior specification, as in (6). Although the alternative specification in (8) delivers invariant and consistency properties, I show in the following proposition that it yields paradoxical inferences.

Paradox 1: *Under the prior specification in* (8)*, the marginal likelihood for any model* M_j *share the same marginalization constant, does not depend on the test assets* R*, and thus achieves test assets irrelevance. However, the marginal likelihood varies with the dimension of the test assets,* N*. Thus, inferences favor one model or another in unanticipated ways, depending on the number of test assets.*

Proof. The proof is straight forward from the matrix determinant lemma.[1] The marginal likelihood of M_j is shown to be proportional to

$$ML_j \propto |F_j^T F_j|^{\frac{(T-N-K_j)}{2}} |RSS_j|^{\frac{-(T-K_j)}{2}}, \tag{11}$$

where RSS_j is the matrix form of the residual sum of squares in the regression of included factors on a constant. It is evident from (11) that ML_j does not depend on R but depends on the number of test assets. □

This paradoxical result is due to the inherent hierarchy in the regressions (1) and (2), and the fact that the test assets, R, and traded factors, F, play distinct roles in the model comparisons. For example, test assets should always be priced by the set of all factors (or the included factors). But the factors need not be priced by the test assets. However, the priors in (8) entertain such economically implausible possibilities by treating test assets and traded factors symmetrically. The independent priors of BS-CZZ do not entertain such possibilities and thus yield valid model comparisons.

To summarize, in the context of comparing models with traded factors, I show that it is *necessary* to induce the priors for the parameters from two independent Jeffreys (1998), as in BS-CZZ.

[1] To conserve space, I have not included the algebraic computations. However, they are available upon request.

II. Priors for Testing an Individual Model with Non-Traded Factors

Notations: r_t denotes the $N \times 1$ vector of excess returns of N test assets at time t, f_t^N is the $K \times 1$ vector of K non-traded factors at time t, f_t^m is the $K \times 1$ vector of mimicking portfolios for the K non-traded factors, $R_{(T\times N)}$ is the $(T \times N)$ matrix form of excess returns for the N test assets over T periods, where each t^{th} row equals $r_t^{'}$, $F^N_{(T\times K)}$ is the $(T \times K)$ matrix form of K non-traded factors over T periods, where each t^{th} row equals $f_t^{N^{'}}$, and $F^m_{(T\times K)}$ is the $(T \times K)$ matrix of mimicking portfolios for the K non-traded factors over T periods, where each t^{th} row equals $f_1^{m'}$.

Recall from the main section of the paper that testing an individual model containing non-traded factors involves three regressions given by

$$F^N_{T\times K} = 1_{T\times 1}c^T + R_{T\times N}W_{N\times K} + \eta_{T\times K},\ \eta \sim MVN(0, \Sigma_\eta \otimes I), \tag{12}$$

$$\bar{R}_{T\times(N-K)} = 1\bar{\alpha}^T + RW\bar{B}_{K\times(N-K)} + \bar{\epsilon}_{T\times(N-K)},\ \bar{\epsilon} \sim MVN(0, \bar{\Sigma}_\epsilon \otimes I), \tag{13}$$

$$RW|W = 1\alpha^T_{mim} + \epsilon_{mim},\ \epsilon_{mim} \sim MVN(0, \Sigma_{mim} \otimes I) \tag{14}$$

Also, recall that I induce the priors for these parameters using two independent Jeffreys (1998) that correspond to the regressions

$$F^N_{T\times K} = 1_{T\times 1}c^T + R_{T\times N}W_{N\times K} + \eta_{T\times K},\ \eta \sim MVN(0, \Sigma_\eta \otimes I), \tag{15}$$

$$R = \mu_R + \epsilon_R,\ \ \sim MVN(0, \Sigma_R \otimes I) \tag{16}$$

The resultant priors are given by

$$P(C, W, \Sigma_\eta) \propto |\Sigma_\eta|^{-(K+1)/2};\ p(\mu_R, \Sigma_R) \propto |\Sigma_R|^{-(N+1)/2}. \tag{17}$$

I have shown in the main section of the paper that these priors yield valid, invariant inferences.

Rather than considering the prior specification in (17), one could consider inducing the priors using a joint regression of non-traded factors and test-asset returns given by

$$\begin{bmatrix} F^N \\ R \end{bmatrix} = \begin{bmatrix} 1\mu_N^T \\ 1\mu_R^T \end{bmatrix} + \epsilon_{NR},\ \epsilon_{NR} \sim MVN(0, \Sigma_{NR} \otimes I), \tag{18}$$

where the priors are given by

$$P\left(\mu_N, \mu_R, \Sigma_{NR}\right) = |\Sigma_{NR}|^{-\frac{N+K}{2}} \tag{19}$$

As in the previous section, it turns out the priors in (19) yield marginal likelihoods that are invariant to the choice of subset of test-assets, $\bar{R}$. However, I show in the following proposition that these priors deliver intuitively paradoxical inferences.

***Paradox* 2**: *The induced priors for the test-asset return covaraince using (19) varies with the number of non-traded factors, K, and thus yield unanticipated inferences, depending on K.*

Proof. Applying the matrix determinant lemma, I get the following induced priors using (19)

$$P(c, W, \Sigma_\eta) \propto |\Sigma_\eta|^{-(N+K+1)/2};\ \Sigma_R \propto |\Sigma_R|^{-(N-K+1)/2} \tag{20}$$

Note that the exponent of $|\Sigma_R|$ includes K. The dependence of return covaraince on the number of non-traded factors is economically counterintuitive. Whereas theory advocates that the expected returns of the test assets should relate to the K factors, it does not impose any apriori restriction on the (cross-sectional) covariance structure of the returns. □

This paradoxical result is again due to the inherent hierarchy in the regressions (15) and (16). For example, the mimicking portfolios should always be obtained by regressing F^N on R, but not, say, R on F^N. The priors in (20) entertain such possibilities. However, the independent Jeffreys (1998) priors that I use in the main section of the paper, also given in (17), do not entertain such possibilities, and thus yield valid inferences.

To summarize, in the context of testing an individual model containing non-traded factors, it is *necessary* to treat non-traded factors and test assets asymmetrically by specifying the priors with two independent Jeffreys (1998).

III. Priors for Comparing Models with Non-Traded Factors

Notations: Let F^N be the set of all non-traded factors. Let R be the excess returns of N test assets, K^T and K^N be the total number of traded factors and non-traded factors across all the models, respectively. F, F^m denote the set of all traded factors and mimicking portfolios across the models, respectively.

Recall from the main section that, to compare models containing non-traded factors, I induce the priors for the parameters from three independent Jeffreys (1998) that correspond to the following regressions

$$F^N = c + [R, F]W + \eta,\ \eta \sim MVN(0, \Sigma_\eta), \tag{21}$$

$$R = [F, F^m]\beta_r + \epsilon_r, \epsilon_r \sim MVN(0, \Sigma_r \otimes I), \tag{22}$$

$$[F, F^m] = 1\alpha^T_{F,F^m} + \epsilon_{F,F^m}, \epsilon_{F,F^m} \sim MVN(0, \Sigma_{F,F^m} \otimes I) \tag{23}$$

As in the previous sections, one may consider inducing the priors from the following alternative specifications.

Alternative Specification-1: In this specification, priors are induced from a single Jeffreys (1998) that corresponds to the joint regression of non-traded factors, test assets and traded factors given by

$$\begin{bmatrix} F^N \\ R \\ F \end{bmatrix} = \begin{bmatrix} 1\mu_N^T \\ 1\mu_R^T \\ 1\mu_F^T \end{bmatrix} + \epsilon_{NR},\ \epsilon_{NR} \sim MVN(0, \Sigma_{NR} \otimes I), \tag{24}$$

where the priors for the parameters are given by

$$P\left(\mu_N, \mu_R, \mu_F, \Sigma_{NR}\right) = |\Sigma_{NR}|^{-\frac{N+K^T+K^N}{2}}. \tag{25}$$

Alternative Specification-2: In this specification, priors are induced from two independent Jeffreys (1998) that correspond to the following regressions

$$F^N_{T\times K} = 1_{T\times 1}c^T + R_{T\times N}W_{N\times K} + \eta_{T\times K},\ \eta \sim MVN(0, \Sigma_\eta \otimes I), \tag{26}$$

$$\begin{bmatrix} R \\ F \end{bmatrix} = \begin{bmatrix} 1\mu_R^T \\ 1\mu_F^T \end{bmatrix} + \epsilon_{RF},\ \epsilon_{RF} \sim MVN(0, \Sigma_{RF} \otimes I), \tag{27}$$

where the priors for the parameters are given by

$$P(C, W, \Sigma_\eta) \propto |\Sigma_\eta|^{-(K^N+1)/2};\ p(\mu_R, \mu_f, \Sigma_R) \propto |\Sigma_R|^{-(N+K^T+1)/2}. \tag{28}$$

As in the previous sections, it turns out that both the alternative specifications in (25) and (28) yield invariant model comparisons. However, conditional on the W, the priors in (25) and in (28) yield paradoxical inferences. Paradox-2 and Paradox-1 prevail under the priors (25) and (28), respectively.

Again, these paradoxical results are due to the inherent hierarchy in the regressions (21), (22) and (23). The mimicking portfolios are always obtained by regressing F^N on $\{R, F\}$, but not, say, $\{R, F\}$ on F^N. Similarly, the test assets R should always be priced by the set of all traded factors and mimicking portfolios, $\{F, F^m\}$, but factors need not always be priced by the test assets. The alternative prior specifications entertain these economically implausible scenarios, thereby yielding paradoxical inferences. However, the prior specification that I use in the main section of the paper do not entertain such possibilities, and thus yield valid inferences.

To summarize, in the context of comparing models containing non-traded factors, it is *necessary* to treat non-traded factors, test assets, traded factors and mimicking portfolios asymmetrically by specifying the parameters with three independent Jeffreys (1998).

References

Barillas, Francisco, and Jay Shanken, 2020, Comparing Priors for Comparing Asset Pricing Models, *Forthcoming, Comments and Rejoinder, The Journal of Finance* .

Chib, Siddhartha, Xiaming Zeng, and Lingxiao Zhao, 2020, On Comparing Asset Pricing Models, *The Journal of Finance* 75, 551–577.

Jeffreys, Harold, 1998, *The Theory of Probability* (OUP Oxford).